Math 261B Thurs., Dec. 3

- Rep. $\mathcal{O}_q$ of $\mathcal{U}_q(\mathfrak{g})$
- $\mathcal{O}_q(G)$
- Canonical bases

X, X^+, α_i, $\alpha_i^\vee$ for G, d_i

Representations of $\mathcal{U}_q(\mathfrak{g})$: finite dimensional standard reps V

$V = \bigoplus_{\lambda \in X} V_\lambda$ $\quad$ V_λ is an $\mathcal{O}_q(T)$-comodule

$K^\beta \curvearrowright V_\lambda$ as $q^{\langle \beta, \lambda \rangle}$

- Classification:

$\lambda \in X^+$ $\quad (\langle \alpha_i^\vee, \lambda \rangle \geq 0 \;\; \forall i)$

There is unique irr. standard fin. dim $V = V^\lambda$ gen. by a $\mathbb{Q}(q)$ h.w. vector v_λ of weight λ:

$$\left.\begin{aligned} &E_i v_\lambda = 0 \quad \forall i \\ &\mathcal{U}_q(\mathfrak{sl}_2) \text{ reps} \Rightarrow \quad F_i^{\langle \alpha_i^\vee, \lambda \rangle + 1} = 0 \end{aligned}\right\} \text{ presentation of } V^\lambda$$

- Quantum vs. classical:

(1) The character $\chi^\lambda(x) = \sum_\mu (\dim V^\lambda_\mu)\, x^\mu$ is the same as for classical V^λ:

$$\chi^\lambda = \sum_{w \in W} w\left(\frac{x^\lambda}{\prod_{\alpha \in R_+} (1 - x^{-\alpha})} \right)$$

(2) $A = \mathbb{Q}[q^{\pm 1}]$: Let V^λ_A be the A-submodule of V^λ gen. by all $F_{i_1}^{(k_1)} \cdots F_{i_\ell}^{(k_\ell)} v_\lambda$

$$F_i^{(k)} = F_i^k / ((k)_{q_i}!) \qquad q_i = q^{d_i}$$

$$(k)_q = \frac{q^k - q^{-k}}{q - q^{-1}}$$

Also closed under $E_i^{(k)}$'s, weight graded.

V_A^λ is a f.g., free A-module.

In any A basis of V_A^λ, $E_i^{(k)}, F_i^{(k)}$ act with coefficients in $\mathbb{Q}[q^{\pm 1}]$

$$V_A^\lambda \otimes_A \underbrace{(A/(q-1))}_{\cong\ \mathbb{Q}} \cong V_{\mathbb{Q}}^\lambda \quad \text{classical}$$

(with $A = \mathbb{Z}[q^{\pm 1}]$, $V_A^\lambda \otimes_A (A/(q-1)) \cong V_{\mathbb{Z}}^\lambda$)

– Complete reducibility : every fin dim, standard V is a $\bigoplus$ of (V^λ)'s.

$\mathbb{Q}(q)$

$\mathcal{O}_q(G) \underset{\text{def}}{=}$ subalg. of $U_q(\mathfrak{g})^*$
gen by all matrix coeff's of all V^λ.
(spanned)

$\mathcal{O}_q(G) \underset{\text{vector space}}{\cong} \bigoplus V_\lambda \otimes V_\lambda^*$ as a left + right $U_q(\mathfrak{g})$ module
(Peter-Weyl)

$A = \mathbb{Q}[q^{\pm 1}]$: use V_A^λ, A-subalg. of $U_q(\mathfrak{g})^*$
$\mathcal{O}_A(G)$

$\mathcal{O}_q(G) = \mathbb{Q}(q) \otimes_A \mathcal{O}_A(G)$

$\mathcal{O}_{\mathbb{Q}}(G) \cong \underbrace{(A/(q-1))}_{= \mathbb{Q}} \otimes_A \mathcal{O}_A(G)$ (can get $\mathcal{O}_\varepsilon(G)$ too)

$\mathcal{O}_q(G)$ is "more natural" than $U_q(\mathfrak{g})$ —
reduces to $\mathcal{O}(G)$ at $q=1$ easily

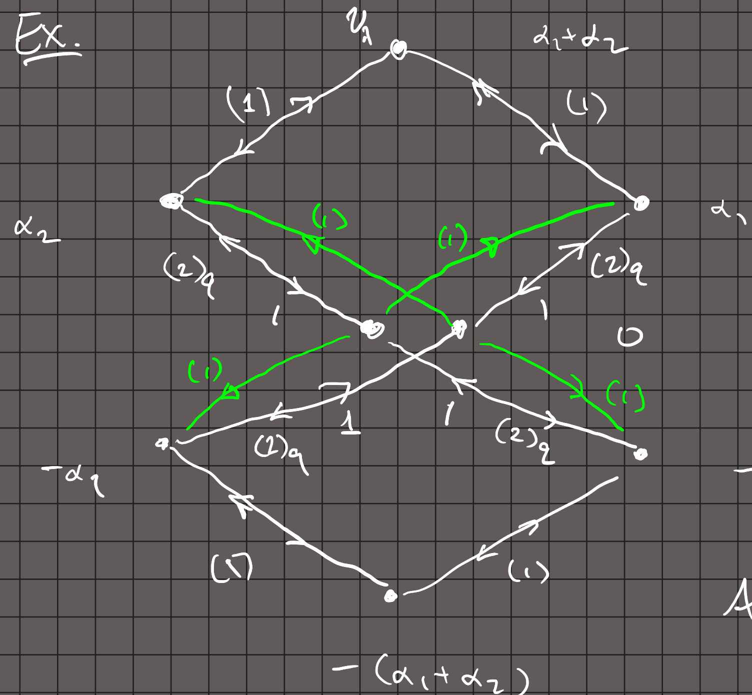
— $\mathcal{O}_q(G)$ depends on d_i only via $q_i = q^{d_i}$

$d_i \to m d_i$, just adjoins $q^{1/m}$

$U_q(G)$ more complicated b/c $K_i = K^{d_i d_i^\vee}$

Canonical basis / Crystal basis (Lusztig / Kashiwara)

Ex.



$G = Sl_3$ $\quad V^\lambda$ $\quad \lambda = (1,0,-1)$

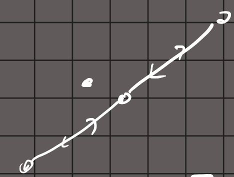
$= \alpha_1 + \alpha_2$

$$\begin{array}{ccc} E_2 & & E_1 \\ & \nwarrow \; \nearrow & \\ & \swarrow \; \searrow & \\ F_1 & & F_2 \end{array}$$

$d_1 = d_2 = 1$

$q_i = q$

$(2)_q = q + q^{-1}$



All coefficients are in $\mathbb{Z}[q+q^{-1}]$

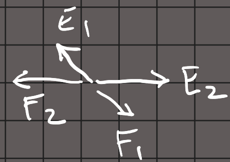
Features of this basis:

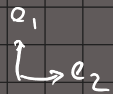
(1) Combinatorially, it decomposes into 'root strings' with matrix coefficients along strings are the standard $U_{q_i}(\mathfrak{sl}_2)_i$ ones.

(2) Non-zero "off-string" coefficients go from shorter strings to longer strings <u>and</u> have degree (in $q+q^{-1}$) strictly bounded by the degree of the on-string coefficient into the target.

V^λ $\quad \lambda = (2,0)$

$G = Sp_4 \quad X = X^* = \mathbb{Z}^2$





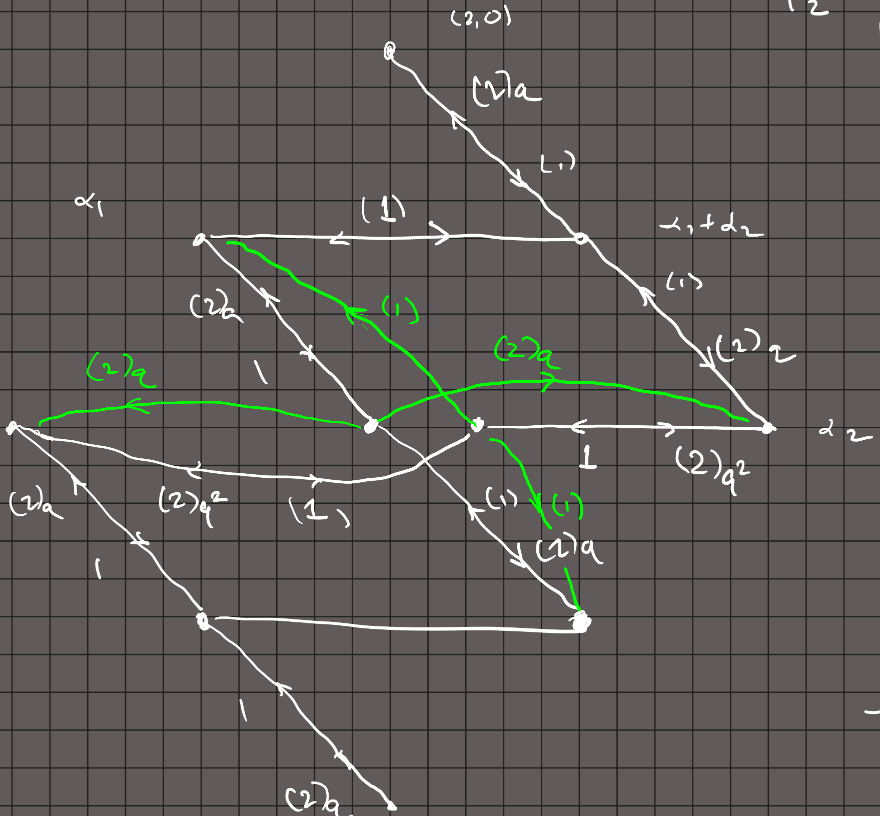
$\alpha_1 = e_1 - e_2 \qquad \alpha_2 = 2e_2$

$\alpha_1^\vee = \varepsilon_1 - \varepsilon_2 \qquad \alpha_2^\vee = \varepsilon_2$

$\langle \alpha_1^\vee, \alpha_2 \rangle = -2 \qquad \langle \alpha_2^\vee, \alpha_1 \rangle = -1$

$d_1 = 1 \qquad d_2 = 2$

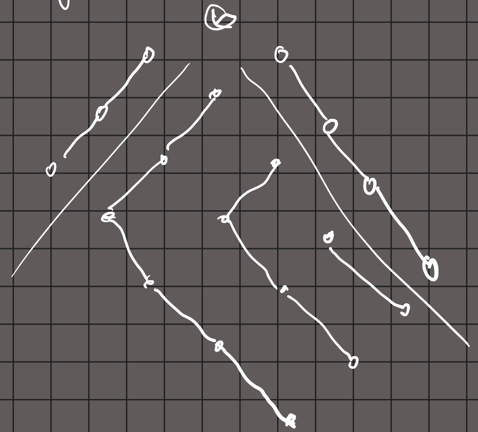
$q_1 = q \qquad q_2 = q^2$



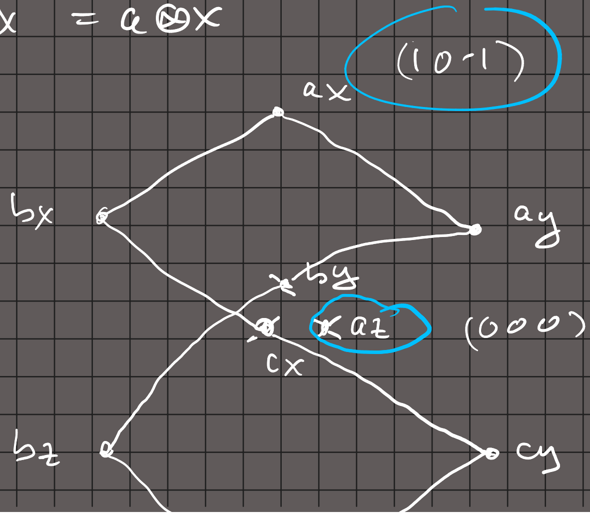
$(2)_{q^2} = q^2 + q^{-2}$

- Exists + unique!
- At $q=1$ it becomes a Chevalley basis.
- In types A, D, E all coefficients are in $\mathbb{N}[q^{\pm 1}]$

Tensor product



$ax = a \otimes x$



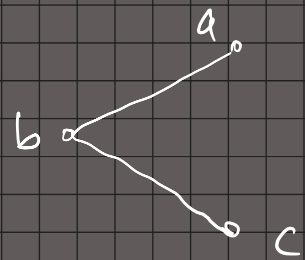
Conjectured true in all types

(Kac-Moody)

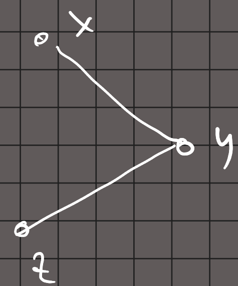
"Categorification"

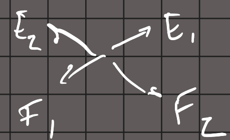
sl_3

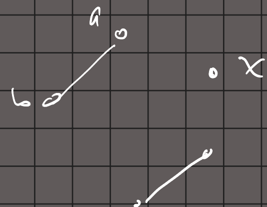
$\lambda = (1,0,0)$

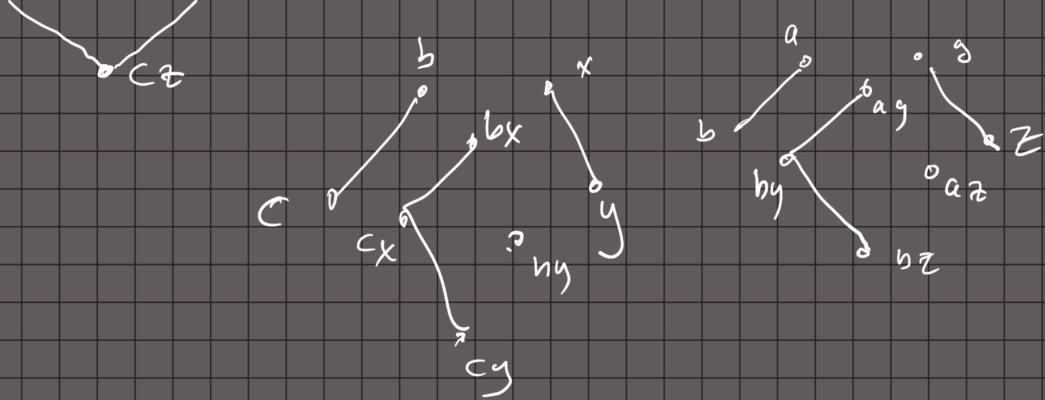


$\lambda = (0\,0\,{-1})$





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